Mid-Semestral Examination : Differential Topology. BMath II

Max. Marks : 30 Time : 2 hours 30 minutes

Answer all questions. You may use theorems/propositions proved in the class after correctly stating them. Any other claim must be accompanied by a proof.

(1) Decide whether the following statements are *True* or *False*. Justify. Answers without correct and complete justifications will not be awarded any marks.

(a) Given $p = (x, y, z) \in S^2$, there exists a smooth function $f : S^2 \longrightarrow S^1$ that is a submersion at both p and $-p = (-x, -y, -z)$.

(b) The map $f : GL_3(\mathbb{R}) \longrightarrow GL_3(\mathbb{R})$ defined by $f(A) = A^2$ is a submersion at

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(c) Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the map defined by $f(x, y, z) = (xy, yz)$. Then $f \pitchfork S^1$.

(d) The hyperboloid defined by $x^2+y^2-z^2 = 1$ (in $\mathbb{R}^3$) intersects the sphere $x^2+y^2+z^2 = a$ (in $\mathbb{R}^3$) transversally for all $a \geq 0$. [3+3+3+3]

(2) Let $g : GL_n(\mathbb{R}) \longrightarrow GL_n(\mathbb{R})$ be the map defined by $g(A) = A^{-1}$. Give a convincing argument (no rigorous proof is expected) that g is smooth. Compute the derivative dg_A of g at $A \in GL_n(\mathbb{R})$. [1+5]

(3) Let I_n denote the $n \times n$ identity matrix and Ω denote the $(n + 1) \times (n + 1)$ block matrix

$$\Omega = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix}$$

Let $X = \{A \in M_{n+1}(\mathbb{R}) : A^t\Omega A = \Omega\}$. Show that X is a manifold. Find $\dim(X)$. [5+1]

(4) Let $f : X \longrightarrow Y$ be a smooth map that is one-one on a compact submanifold Z of X. Suppose that for all $x \in Z$, the derivative $df_x : T_x(X) \longrightarrow T_{f(x)}(Y)$ is an isomorphism. Show that $f : Z \longrightarrow f(Z)$ is a diffeomorphism. Further show that there exists an open set U in X with $Z \subseteq U$ such that $f(U)$ is open in Y and

$$f : U \longrightarrow f(U)$$

a diffeomorphism. [1+5]